First-Order Linear Homogeneous Systems of ODEs Linear Algebra X. Du

- Takes the form $\vec{x}' = A\vec{x}$, where $\vec{x}$ is a vector, and A is a matrix
- Example: $x' = x + y$, $y' = x - y$ can be represented as:
  - $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.
- How to solve:
  - Method 1: elimination (this should give a higher order ODE).
    - Elimination also works for non-linear systems
  - Method 2 (the better one): eigenvalue and eigenvector analysis
- Solving systems of ODEs with eigenvalues and eigenvectors
  - Justification:
    - Guess $\vec{x} = \vec{v}e^{\lambda t}$. $\vec{x}' = A\vec{x} \Rightarrow \lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t}$ $\lambda\vec{v} = A\vec{v}$
    - Therefore, our solution takes the form $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an eigenvector of A and λ is its associated eigenvalue.
  - Steps:
    - 1. Find all eigenvectors and associated eigenvalues of A.
      - If A is a defective matrix, then find the generalized eigenvectors.
    - 2. Apply superposition: $\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} + \dots + c_n\vec{v}_ne^{\lambda_n t}$ is the general solution given a first-order linear homogeneous system of n ODEs.
      - If the eigenvalues and eigenvectors are complex, find the solution $\vec{x}$ normally, then apply the theorem that $\mathrm{Re}[\vec{x}]$ and $\mathrm{Im}[\vec{x}]$ also solve the system to find distinct linearly independent solutions.
  - Eigenvectors also make up the columns of the inverse of a decoupling matrix. Decoupling uses a change of variables to find a system of independent ODEs.
- **Phase Portrait** (usually for systems of two ODEs)**:** plot of these solutions as a set of parametric curves.
  - Real distinct eigenvalues produce phase portraits with nodal source (if $\lambda_1 > \lambda_2 > 0$), nodal sink (if $\lambda_1 < \lambda_2 < 0$), or saddle (if $\lambda_1 > 0 > \lambda_2$)
  - Complex eigenvalues produce spiral source (if $\mathrm{Re}[\lambda_1] > 0$), spiral sink (if $\mathrm{Re}[\lambda_1] < 0$), or ellipses (if $\mathrm{Re}[\lambda_1] = 0$).
  - Weird phase portraits occur when there are repeated eigenvalues
- **Fundamental Matrix**: For a first-order system of n linear homogeneous ODEs, let $\vec{x}_1$, $\vec{x}_2$,…, $\vec{x}_n$ all be column vectors, each with n components, representing linearly independent solutions of $\vec{x}' = A\vec{x}$. The fundamental matrix is $\Psi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$.
  - The general solution of $\vec{x}' = A\vec{x}$ is $\vec{x} = \Psi\vec{c}$, where $\vec{c}$ is a constant vector.
  - $\Psi' = A\Psi$ as each column is a solution
- **Wronskian Determinant** for systems:
  - For a system of two linear homogeneous ODEs:
    - $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 \end{vmatrix}$ Note: $\vec{x}_1$, $\vec{x}_2$ are column vectors with 2 components
  - For a system of n linear homogeneous ODEs:
  - $W(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{vmatrix} = \det(\Psi)$ Usually, $W \neq 0$.